

On estimation states of hidden markov models in condition of unknown transition matrix

Vasily Vasilyev, Alexander Dobrovidov

Abstract—In this paper, we develop methods of nonlinear filtering and prediction of an unobservable Markov chain with a finite set of states. This Markov chain controls coefficients of AR(p) model. Using observations generated by AR(p) model we have to estimate the state of Markov chain in the case of an unknown probability transition matrix. Comparison of proposed non-parametric algorithms with the optimal methods in the case of the known transition matrix is carried out by simulating.

Index Terms—hidden markov models, statistical signal processing, filtering and prediction, optimization problem, kernel density estimation.

I. INTRODUCTION

HIDDEN markov models are very popular for modeling and simulating processes, when you do not observe...

II. SYSTEM MODEL

Let (S_n, X_n) be a two-component process, where (S_n) is unobservable component and (X_n) is observable one, $n \in \{1, 2, \dots, N\}$, $N \in \mathbf{N}$; (S_n) “controls” equation coefficients of (X_n) . Let (S_n) be a stationary Markov chain with M discrete states and transition matrix $\|p_{i,j}\|$, $p_{i,j} = \Pr(S_n = j \mid S_{n-1} = i)$. The process (X_n) is described by the autoregressive model of order p :

$$X_n = \mu(S_n) + \sum_{i=1}^p a_i(S_n)(X_{n-i} - \mu(S_n)) + b(S_n)\xi_n, \quad (1)$$

where $\{\xi_n\}$ are i.i.d. random variables with the standard normal distribution, $\mu, a_i, b \in \mathbf{R}$ are coefficients controlled by the process (S_n) .

As a quality measure for our methods we use mean risk $E(L(S_n, \hat{S}_n))$ with a simple loss function L :

$$L(S_n, \hat{S}_n) = \begin{cases} 1, & S_n \neq \hat{S}_n, \\ 0, & S_n = \hat{S}_n, \end{cases} \quad (2)$$

where $\hat{S}_n = \hat{S}_n(X_1^n)$ is an estimator of S_n and $X_1^n = (X_1, X_2, \dots, X_n)$.

As known, for this risk function with the loss function (2) the optimal estimator is

$$\hat{S}_n = \underset{m \in \{1, \dots, M\}}{\operatorname{argmax}} \Pr(S_n = m \mid X_1^n), \quad (3)$$

where $\Pr(S_n = m \mid X_1^n)$ is a posterior probability with respect to a σ -algebra, generated by r.v. X_1^n . Its realization will be denoted by

$$P(S_n = m \mid X_1^n = x_1^n) = P(S_n = m \mid x_1^n), \quad (4)$$

where we will write x_1^n instead of $X_1^n = x_1^n$.

A. Basic equations

In this paper we consider methods of filtering and prediction in the case of unknown parameters (transition matrix) of process (S_n) and known parameters (equation coefficients in (1)) of process (X_n) . For comparison with some standard we also consider optimal filtering and prediction, where all parameters are known.

Filtering is a problem to estimate S_n by using X_1^n . Therefore basic equations for filtering

$$\begin{aligned} P(S_n = m \mid x_1^n) \\ = \frac{f(x_n \mid S_n = m, x_1^{n-1})}{f(x_n \mid x_1^{n-1})} P(S_n = m \mid x_1^{n-1}), \end{aligned} \quad (5)$$

$$\begin{aligned} f(x_n \mid x_1^{n-1}) \\ = \sum_{m=1}^M f(x_n \mid S_n = m, x_1^{n-1}) P(S_n = m \mid x_1^{n-1}), \end{aligned} \quad (6)$$

can be obtained from the total probability formula. Since coefficients in (1) are known and $\xi_n \sim \mathcal{N}(0, 1)$ then

$$\begin{aligned} f(x_n \mid S_n = m, x_1^{n-1}) \\ = f(x_n \mid S_n = m, x_{n-p}^{n-1}) = f_m(x_n), \end{aligned} \quad (7)$$

where

$$\begin{aligned} f_m(x_n) \\ = \phi\left(x_n; \mu(m) + \sum_{i=1}^p a_i(m)(x_{n-i} - \mu(m)), b^2(m)\right) \end{aligned} \quad (8)$$

with normal probability density function

$$\phi(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right), \quad (9)$$

where $x, \mu \in \mathbf{R}$, $\sigma \in \mathbf{R}^+$.

V. Vasilyev is with Department of Radio Engineering and Cybernetics, Moscow Institute of Physics and Technology (State University), Moscow, Russia e-mail: evil.vasy@gmail.com

A. Dobrovidov is with Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia email: dobrovidov@gmail.com

Manuscript received ***** **, 2015; revised ***** **, 2015.

III. OPTIMAL FILTERING

In the optimal filtering all parametres are known. We use (7) knowing coefficients in (1) and calculate $P(S_n = m | x_1^{n-1})$ in (5) knowing transition matrix:

$$P(S_n = m | x_1^{n-1}) = \sum_{i=1}^M p_{i,m} P(S_{n-1} = i | x_1^{n-1}). \quad (10)$$

Then the (5) is transformed to the evaluation equation [1]

$$P(S_n = m | x_1^n) = \frac{f_m(x_n) \sum_{i=1}^M p_{i,m} P(S_{n-1} = i | x_1^{n-1})}{\sum_{j=1}^M f_j(x_n) \sum_{i=1}^M p_{i,j} P(S_{n-1} = i | x_1^{n-1})},$$

which will be considered as the optimal standard.

IV. NON-PARAMETRIC FILTERING

A. Reducing to optimization problem

In this section, the transition matrix $\|p_{i,j}\|$ is assumed unknown, therefore we can not use the equation (10). To overcome this uncertainty we include formula (7) in equations (5), (6) and obtain

$$P(S_n = m | x_1^n) = \frac{f_m(x_n)}{f(x_n | x_1^{n-1})} u_n(m), \quad (11)$$

$$f(x_n | x_1^{n-1}) = \sum_{m=1}^M f_m(x_n) u_n(m), \quad (12)$$

where

$$u_n(m) = P(S_n = m | x_1^{n-1}), \quad \forall m = 1, \dots, M$$

are new variables, which do not depend on x_n and

$$\sum_{i=1}^M u_i = 1, \quad u_m \geq 0, \quad \forall m = 1, \dots, M.$$

To calculate (11) and (12) it is necessary to find all $u_n(m)$. We need to make the assumption. We suppose that process (S_n, X_n) is α -mixing, then

$$f(x_n | x_1^{n-1}) \approx f(x_n | x_{n-\tau}^{n-1}), \quad \tau \in \{1, 2, \dots, n-1\},$$

and estimate density $f(x_n | x_{n-\tau}^{n-1})$ using kernel density estimation and designate this estimator like $\hat{f}(x_n | x_{n-\tau}^{n-1})$.

Let us introduce vector $\mathbf{u}_n = (u_n(1), u_n(2), \dots, u_n(M))$ with unknown elements $u_n(m), m = 1, \dots, M$. Then for calculating \mathbf{u}_n one proposes the following estimator

$$\hat{\mathbf{u}}_n = \underset{\mathbf{u} \in \Delta_M}{\operatorname{argmin}} \int_{-\infty}^{+\infty} |\hat{f}(z_n | x_{n-\tau}^{n-1}) - \sum_{m=1}^M f_m(z_n) u_m|^2 dz_n, \quad (13)$$

where

$$\Delta_M = \left\{ (t_1, t_2, \dots, t_M) \in \mathbf{R}^M \mid \sum_{i=1}^M t_i = 1, t_i \geq 0, \forall i \in \{1, 2, \dots, M\} \right\}$$

is simplex. Let us rewrite estimator $\hat{\mathbf{u}}_n$ with more details:

$$\hat{\mathbf{u}}_n = \underset{\mathbf{u} \in \Delta_M}{\operatorname{argmin}} I_1 - 2I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} \hat{f}^2(z_n | x_{n-\tau}^{n-1}) dz_n, \\ I_2 &= \int_{-\infty}^{+\infty} \sum_{m=1}^M \hat{f}(z_n | x_{n-\tau}^{n-1}) f_m(z_n) u_m dz_n, \\ I_3 &= \int_{-\infty}^{+\infty} \sum_{i=1}^M \sum_{j=1}^M f_i(z_n) f_j(z_n) u_i u_j dz_n. \end{aligned}$$

Since I_1 does not depend on \mathbf{u} , then reduce it, also transform I_2 and I_3 , so $\hat{\mathbf{u}}_n$ has representation

$$\begin{aligned} \hat{\mathbf{u}}_n &= \underset{\mathbf{u} \in \Delta_M}{\operatorname{argmin}} I_3 - 2I_2 \\ &= \underset{\mathbf{u} \in \Delta_M}{\operatorname{argmin}} \sum_{i=1}^M \sum_{j=1}^M c_{ij} u_i u_j - 2 \sum_{m=1}^M c_m u_m, \end{aligned} \quad (14)$$

where

$$c_{ij} = \int_{-\infty}^{+\infty} f_i(z_n) f_j(z_n) dz_n, \quad (15)$$

$$c_m = \int_{-\infty}^{+\infty} \hat{f}(z_n | x_{n-\tau}^{n-1}) f_m(z_n) dz_n. \quad (16)$$

To solve optimization problem (14), primarily, it is necessary to calculate latter coefficients (15) and (16), which we will obtain using kernel density estimators. Therefore we introduce following chapter.

B. Kernel density estimators

In the general case kernel density estimator of density f is

$$\hat{f}(\mathbf{y}; \mathbf{H}) = \frac{1}{N} \sum_{i=1}^N K_{\mathbf{H}}(\mathbf{y} - \mathbf{Y}_i), \quad (17)$$

where $\mathbf{y} = (y_1, y_2, \dots, y_d)^T$ is argument and $\mathbf{Y}_i = (Y_{i1}, Y_{i2}, \dots, Y_{id})^T$, $i = 1, 2, \dots, N$ are drawn from density f ; $K_{\mathbf{H}}(\mathbf{y}) = |\mathbf{H}|^{-1/2} K(\mathbf{H}^{-1/2} \mathbf{y})$, where $K(\mathbf{y})$ is the multivariate kernel, which is probability density function; $\mathbf{H} \in \mathcal{H}$ is the bandwidth matrix and \mathcal{H} is the set of $d \times d$, symmetric and positive-definite matrixes. We propose to use unbiased cross-validation (UCV) to find \mathbf{H} (univariate case proposed in [2], [3] and multivariate in [4], [5]). This is a popular and relevant method is aimed to estimate

$$\operatorname{ISE}(\mathbf{H}) = \int_{\mathbf{R}^d} (\hat{f}(\mathbf{y}; \mathbf{H}) - f(\mathbf{y}))^2 d\mathbf{y}$$

and then minimize resulting function

$$\begin{aligned} \text{UCV}(\mathbf{H}) &= \frac{1}{N(N-1)} \sum_{i=1}^N \sum_{\substack{j=1, \\ j \neq i}}^N (K_{\mathbf{H}} * K_{\mathbf{H}} - 2K_{\mathbf{H}})(\mathbf{Y}_i - \mathbf{Y}_j) \\ &\quad + \frac{1}{N} R(K) |\mathbf{H}|^{-1/2}, \quad (18) \end{aligned}$$

$$R(K) = \int_{\mathbf{R}^d} K(\mathbf{y})^2 d\mathbf{y},$$

where $*$ denotes a convolution. Then the estimator of \mathbf{H} is

$$\mathbf{H}_{\text{UCV}} = \underset{\mathbf{H} \in \mathcal{H}}{\text{argmin}} \text{UCV}(\mathbf{H}). \quad (19)$$

We suppose to generate components Y_{ik} of vector \mathbf{Y}_i from univariate sample x_1, x_2, \dots, x_n according to the rule

$$Y_{ik} = x_{(i-1)l+k}, \quad k = 1, 2, \dots, d$$

where $l \in \mathbf{N}$ influences on stochastic dependence between vectors \mathbf{Y}_i (for bigger l less dependence). Then we suggest to simplify obtaining of estimator (17) and function (18). For this aim we:

- use normal kernel, it means that we set equal \mathbf{H} to d -variate normal density with zero mean vector and identity covariance matrix ϕ ;
- use scalar h^2 multiple of identity $d \times d$ matrix (\mathbf{I}_d) for bandwidth matrix:

$$\mathbf{H} = h^2 \mathbf{I}_d.$$

Then the estimator (17) becomes

$$\begin{aligned} \hat{f}(\mathbf{y}; h) &= \frac{1}{N(2\pi)^{d/2} h^d} \sum_{i=1}^N \exp \left(-\frac{\sum_{j=1}^d (y_j - x_{(i-1)l+j})^2}{2h^2} \right), \quad (20) \end{aligned}$$

with $N = 1 + \lfloor \frac{n-d}{l} \rfloor$ and the estimator of h is

$$\hat{h} = \underset{h>0}{\text{argmin}} \text{UCV}(h), \quad (21)$$

$$\begin{aligned} \text{UCV}(h) &= \frac{1}{N(N-1)(2\pi)^{d/2} h^d} \sum_{i=1}^N \sum_{\substack{j=1, \\ j \neq i}}^N \frac{1}{2^{d/2}} e^{-\frac{\Delta x_{ij}}{4h^2}} - 2e^{-\frac{\Delta x_{ij}}{2h^2}} \\ &\quad + \frac{1}{N(4\pi)^{d/2} h^d}, \end{aligned}$$

$$\Delta x_{ij} = \sum_{k=1}^d (x_{(i-1)l+k} - x_{(j-1)l+k})^2.$$

Computing minima analytically is a challenge, so a numerical calculation is popular. The function $\text{UCV}(h)$ often has multiple local minima, therefore more correct way is to use brute-force search to find \hat{h} , however it is a very slow algorithm.

In [6] it was shown that spurious local minima are more likely at too small values of h , so we propose to use golden section search between 0 and h^+ , where

$$h^+ = \left(\frac{4}{N(d+2)} \right)^{\frac{1}{d+4}} \max_{k \in \{1, \dots, d\}} \hat{\sigma}_k,$$

where $\hat{\sigma}_k$ is the sample standard deviation of k -th elements of \mathbf{Y}_i . The parameter h^+ is an oversmoothed bandwidth. If the matrix \mathbf{H} was an unconstrained then

$$\mathbf{H}^+ = \left(\frac{4}{N(d+2)} \right)^{\frac{1}{d+4}} \mathbf{S},$$

where \mathbf{S} is a sample covariance matrix of \mathbf{Y}_i . The matrix \mathbf{H}^+ is oversmoothed bandwidth in the most cases. The latter estimator is proposed in [7]. To calculate \mathbf{H}_{UCV} with unconstrained \mathbf{H} you may use quasi-Newton minimization algorithm like in [5].

C. Calculation of coefficients c_{ij} and c_m

For calculating unknown coefficients c_{ij} and c_m in (14) we use formulas (15) and (16). Observe that for normal probability density function (9) following equation

$$\begin{aligned} &\int_{-\infty}^{+\infty} \phi(x; \mu_1, \sigma_1^2) \phi(x; \mu_2, \sigma_2^2) dx \\ &= \phi(\mu_1; \mu_2, \sigma_1^2 + \sigma_2^2) = \phi(\mu_2; \mu_1, \sigma_1^2 + \sigma_2^2) \end{aligned}$$

is correct, therefore using it and (8) we have

$$\begin{aligned} c_{ij} &= \int_{-\infty}^{+\infty} \phi \left(z_n; \mu(i) + \sum_{k=1}^p a_k(i)(x_{n-k} - \mu(i)), b^2(i) \right) \\ &\quad \cdot \phi \left(z_n; \mu(j) + \sum_{k=1}^p a_k(j)(x_{n-k} - \mu(j)), b^2(j) \right) dz_n \\ &= \phi \left(\mu(i) + \sum_{k=1}^p a_k(i)(x_{n-k} - \mu(i)); \right. \\ &\quad \left. \mu(j) + \sum_{k=1}^p a_k(j)(x_{n-k} - \mu(j)), b^2(i) + b^2(j) \right), \quad (22) \end{aligned}$$

also $c_{ij} = c_{ji} > 0$. For calculating c_m we estimate conditional density $\hat{f}(z_n | x_{n-\tau}^{n-1})$ applying (20):

$$\begin{aligned} \hat{f}(z_n | x_{n-\tau}^{n-1}) &= \frac{\hat{f}(z_n, x_{n-\tau}^{n-1})}{\int_{-\infty}^{+\infty} \hat{f}(z_n, x_{n-\tau}^{n-1}) dz_n} \\ &= \sum_{i=1}^N \beta_{ni}(\tau) \phi(z_n; x_{(i-1)l+\tau+1}, h^2), \end{aligned}$$

$$\beta_{ni}(\tau) = \frac{\exp\left(-\frac{\sum_{j=-\tau}^{-1} (x_{n+j} - x_{(i-1)l+j+\tau+1})^2}{2h^2}\right)}{\sum_{k=1}^N \exp\left(-\frac{\sum_{j=-\tau}^{-1} (x_{n+j} - x_{(k-1)l+j+\tau+1})^2}{2h^2}\right)},$$

where $N = 1 + \lfloor \frac{n-1-d}{l} \rfloor$, bandwidth h is estimated by (21). Remark that $\beta_{ni}(\tau)$ does not depend on z_n . Then we substitute latter estimator in (16) and obtain

$$\begin{aligned} c_m &= \int_{-\infty}^{+\infty} \hat{f}(z_n | x_{n-\tau}^{n-1}) f_m(z_n) dz_n \\ &= \int_{-\infty}^{+\infty} \sum_{i=1}^N \beta_{ni}(\tau) \phi(z_n; x_{(i-1)l+\tau+1}, h^2) \\ &\quad \cdot \phi\left(z_n; \mu(m) + \sum_{k=1}^p a_k(m)(x_{n-k} - \mu(m)), b^2(m)\right) dz_n \\ &= \sum_{i=1}^N \beta_{ni}(\tau) \int_{-\infty}^{+\infty} \phi(z_n; x_{(i-1)l+\tau+1}, h^2) \\ &\quad \cdot \phi\left(z_n; \mu(m) + \sum_{k=1}^p a_k(m)(x_{n-k} - \mu(m)), b^2(m)\right) dz_n \\ &= \sum_{i=1}^N \beta_{ni}(\tau) \phi\left(x_{(i-1)l+\tau+1}; \right. \\ &\quad \left. \mu(m) + \sum_{k=1}^p a_k(m)(x_{n-k} - \mu(m)), h^2 + b^2(m)\right), \quad (23) \end{aligned}$$

also we remark that $c_m > 0$.

D. Solution of optimization problem

In the previous chapters we reduce main problem to optimization problem

$$\begin{aligned} \hat{\mathbf{u}}_n &= \underset{\mathbf{u} \in \Delta_M}{\operatorname{argmin}} F_n(\mathbf{u}), \\ F_n(\mathbf{u}) &= \sum_{i=1}^M \sum_{j=1}^M c_{ij} u_i u_j - 2 \sum_{m=1}^M c_m u_m, \end{aligned}$$

where coefficients c_{ij} and c_m were calculated in (22) and (23). Let us consider kind of optimization. We have that Δ_m is convex set and Hessian matrix of function $F_n(\mathbf{s})$ is

$$\mathcal{L}''_{\mathbf{u}} = 2 \cdot \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1M} \\ c_{21} & c_{22} & \cdots & c_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ c_{M1} & c_{M2} & \cdots & c_{MM} \end{pmatrix}.$$

If $\mathcal{L}''_{\mathbf{u}}$ is positive defined, then $F_n(\mathbf{s})$ is convex, thus we have convex optimization. In this case we propose to use Karush–Kuhn–Tucker (KKT) conditions [8], [9], because of:

- our case is special because there is opportunity to solve KKT conditions analytically;

- for convex optimization KKT conditions, which are primarily necessary, are also sufficient;

else you may apply methods of quadratic programming. Also we want remark that $\mathcal{L}''_{\mathbf{u}}$ does not depend on variables u_i and coefficients c_m , which means that previous kernel density estimators have no influence on kind of optimization.

Let us consider KKT conditions, then Lagrangian is

$$\mathcal{L} = \lambda_0 F_n(\mathbf{u}) + \sum_{i=1}^M \lambda_i (-u_i) + \lambda_{M+1} \left(\sum_{i=1}^M u_i - 1 \right),$$

where $\lambda^* = (\lambda_0^*, \lambda_1^*, \dots, \lambda_{M+1}^*) \in \mathbf{R}^{M+2}$. We need to find λ^* and \mathbf{u}^* such that stationary condition

$$\begin{aligned} \mathcal{L}'_{u_i} &= 2\lambda_0^* \left(\sum_{j=1}^M c_{ij} u_j^* - c_i \right) - \lambda_i^* + \lambda_{M+1}^* = 0, \\ &\quad \forall i = 1, \dots, M \end{aligned}$$

primal feasibility

$$\begin{aligned} -u_i^* &\leq 0, \quad \forall i = 1, \dots, M \\ \sum_{i=1}^M u_i^* - 1 &= 0, \end{aligned}$$

dual feasibility

$$\lambda_i^* \geq 0, \quad \forall i = 1, \dots, M$$

complementary slackness

$$\lambda_i^* u_i^* = 0, \quad \forall i = 1, \dots, M$$

hold. Let $\lambda_0^* = 0$ to check that the gradients of constraints are linearly independent at \mathbf{u}^* , so KKT conditions lead to system

$$\begin{cases} \lambda_1^* = \lambda_2^* = \dots = \lambda_{M+1}^*, \\ \lambda_i^* u_i^* = 0, \quad \lambda_i^* \geq 0, & \forall i = 1, \dots, M \\ \sum_{i=1}^M u_i^* = 1, \quad u_i^* \geq 0, & \forall i = 1, \dots, M \end{cases}$$

which could be solved only with $\lambda^* = \vec{0}$, which means that gradients of constraints are linearly independent for any \mathbf{u}^* . The vector λ^* is defined with an accuracy of $\alpha > 0$, so we define $\lambda_0 = 1/2$, then KKT conditions lead to a system

$$\mathbf{C} \cdot \vec{\rho} = \mathbf{c},$$

where

$$\begin{aligned} \mathbf{C} &= \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1M} & -1 & 0 & \cdots & 0 & 1 \\ c_{21} & c_{22} & \cdots & c_{2M} & 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{M1} & c_{M2} & \cdots & c_{MM} & 0 & 0 & \cdots & -1 & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \\ \vec{\rho} &= \begin{pmatrix} u_1^* \\ \vdots \\ u_M^* \\ \lambda_1^* \\ \vdots \\ \lambda_{M+1}^* \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \\ 1 \end{pmatrix}, \\ \lambda_i^* u_i^* &= 0, \quad \lambda_i^* \geq 0, \quad u_i^* \geq 0. \quad \forall i = 1, \dots, M \end{aligned}$$

TABLE I
SAMPLE MEAN ERRORS

	Filtering error, %	Prediction error, %
Optimal	16.4	26.6
Non-parametric	22.7	37.6

To solve last system it is necessary to consider all combinations of pairs (u_i^*, λ_i^*) , $\forall i = 1, \dots, M$, where u_i^* or λ_i^* is equal to 0 (not both). Total amount of combinations is equal to 2^M . If $u_i^* = 0$ then i -th column in the matrix \mathbf{C} and i -th row in $\bar{\rho}$ are reduced, else $\lambda_i^* = 0$ and $(M + i)$ -th column in the matrix \mathbf{C} and $(M + i)$ -th row in $\bar{\rho}$ are reduced. After choosing zero element in each pair (u_i^*, λ_i^*) , $\forall i = 1, \dots, M$ matrix \mathbf{C} is reduced to an $(M + 1) \times (M + 1)$ -matrix \mathbf{C}_r and $\bar{\rho}$ to $(M + 1) \times 1$ -matrix $\bar{\rho}_r$. Therefore for each combination it is necessary to calculate

$$\bar{\rho}_r = \mathbf{C}_r^{-1} \cdot \mathbf{c}.$$

If the first M elements in $\bar{\rho}_r$ are non-negative then obtained \mathbf{u}^* is a solution ($\hat{\mathbf{u}}_n$) of optimization problem and there is no reason to calculate $\bar{\rho}_r$ for the next combination, because in convex optimization local minima is global minima.

As a result, we substitute estimator $\hat{\mathbf{u}}_n$ in (11) and (12) and problem of non-parametric filtering is solved.

V. ONE-STEP AHEAD PREDICTION

We will consider one-step ahead prediction. Like for filtering we minimize mean risk $E(L(S_n, \hat{S}_n))$ with simple loss function (2). Therefore optimal estimator of S_n is

$$\hat{S}_n = \underset{m \in \{1, \dots, M\}}{\operatorname{argmax}} \Pr(S_n = m \mid X_1^{n-1}).$$

We remark that probability $\Pr(S_n = m \mid X_1^{n-1})$ is already obtained in the considered approaches of filtering: for optimal prediction it is written in (10) and for non-parametric prediction accordingly in (14). It means that we primarily solve problem of one-step ahead prediction and then filtering problem.

VI. EXAMPLE

Let the Markov chain (S_n) has 3 states ($M = 3$) and transition matrix

$$\|p_{i,j}\| = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.05 & 0.9 & 0.05 \\ 0.1 & 0.05 & 0.85 \end{pmatrix}. \quad (24)$$

Sample volume n is changed from 500 to 600. Observable process (X_n) is simulated like AR(2) model with coefficients $\mu \in \{0, 0.5, 1\}$, $a_1 \in \{0.3, 0.2, 0.1\}$, $a_2 \in \{0.2, 0.3, 0.4\}$, $b \in \{0.1, 0.2, 0.1\}$. Also we take $\tau = 2$ and $l = 1$. The results are presented in Fig. 1 and sample mean errors after 50 repeated experiments in Table I.

VII. CONCLUSION

Preparing...

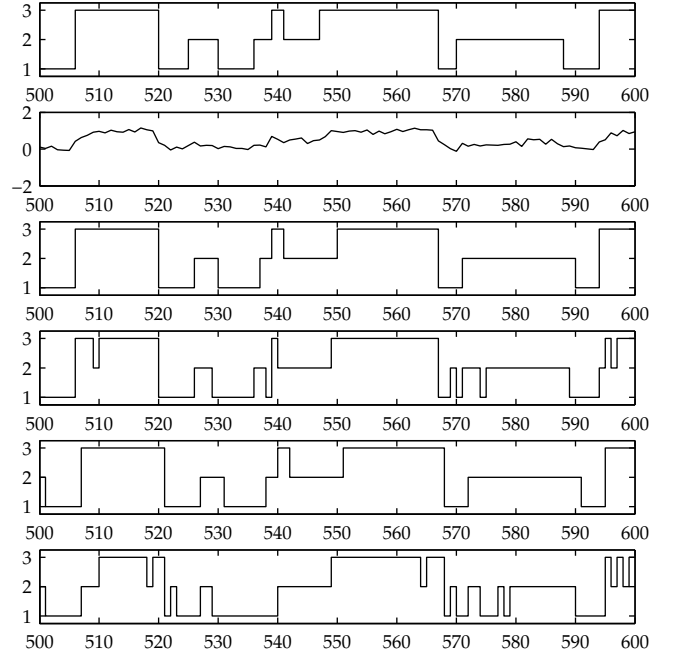


Fig. 1. From top to bottom: 1 — unobservable s_n ; 2 — observable x_n ; 3, 4 — optimal and non-parametric filtering; 5, 6 — optimal and non-parametric prediction.

REFERENCES

- [1] A. V. Dobrovodov, G. M. Koshkin, and V. A. Vasiliev, *Non-parametric models and statistical inference from dependent observations*. USA: Kendrick Press, 2012.
- [2] M. Rudemo, "Empirical choice of histograms and kernel density estimators," *Scandinavian Journal of Statistics*, vol. 9, pp. 65–78, 1982.
- [3] A. Bowman, "An alternative method of cross-validation for the smoothing of density estimates," *Biometrika*, vol. 7, pp. 353–360, 1984.
- [4] S. R. Sain, K. A. Baggerly, and D. W. Scott, "Cross-validation of multivariate densities," *Journal of the American Statistical Association*, vol. 89, pp. 807–817, 1994.
- [5] T. Duong and M. L. Hazelton, "Cross-validation bandwidth matrices for multivariate kernel density estimation," *Scandinavian Journal of Statistics*, vol. 32, no. 3, pp. 485–506, 2005.
- [6] P. Hall and J. Marron, "Local minima in cross-validation functions," *Journal of the Royal Statistical Society, Series B (Methodological)*, vol. 53, pp. 245–252, 1991.
- [7] G. R. Terrell, "The maximal smoothing principle in density estimation," *J. Amer. Statist. Assoc.*, vol. 85, pp. 470–477, 1990.
- [8] H. W. Kuhn and A. W. Tucker, "Nonlinear programming," in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley, California, 1951, pp. 481–492.
- [9] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge: Cambridge University Press, 2004.

Vasily Vasilyev Preparing...

PLACE
PHOTO
HERE



Alexander Dobrovidov Preparing...